Digital Signal Processing[1]

Prof. Brückl

Only for internal use

DFT Examples

- Illustrating the Errors in DFT Processing

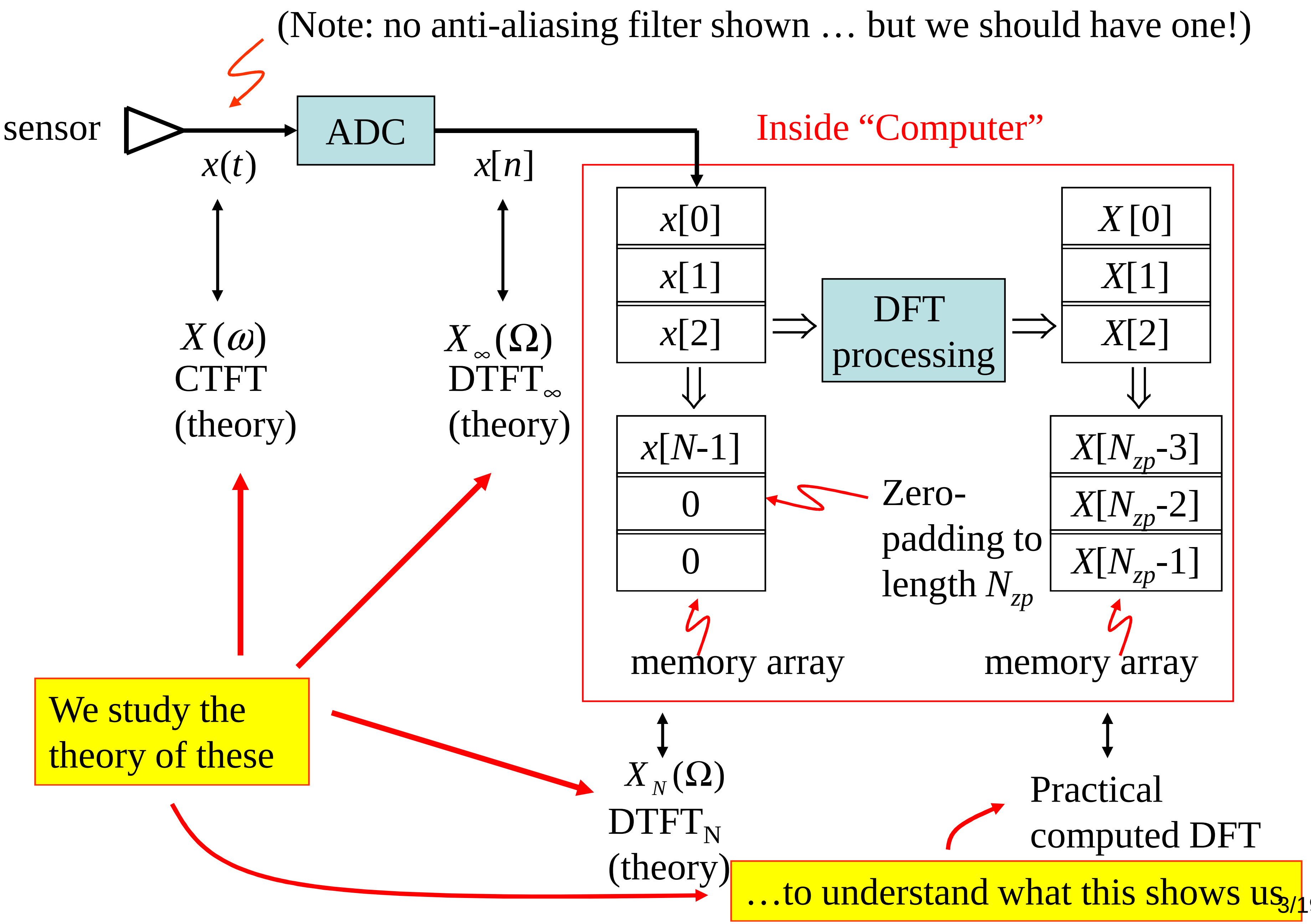
[1]Based on „Fundamtentals of Signals and Systems“, Edward W. Kamen, Bonnie S. Heck, Prentice Hall, Third Edition, courtesy of B. S. Heck and lecture notes „Signals and Systems“ from Prof. M. Fowler, School of Engineering and Applied Science, Binghampton University, State University of New York, courtesy of Fowler, M.

Illustrating the Errors in DFT processing

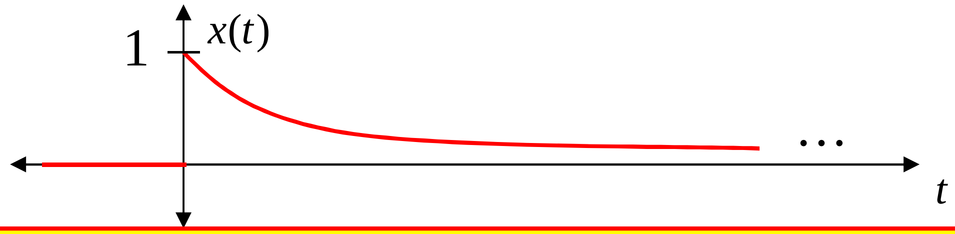
This example (hopefully!) does a nice job of showing the relationships between:

- the CTFT,
- the DTFT of the infinite-duration signal,
- the DTFT of the finite-duration collected samples,
- and the DFT computed from those samples.

Recall the processing setup:



Let's imagine we have the following CT Signal: $x(t) = e^{-bt}u(t) \quad \text{for} \;\; b > 0$



Now … <u>analyze</u> what we <u>will</u> get from the DFT processing for this signal …

From our FT Table we find the FT of $x(t)$ is:

A $\quad \boxed{X(\omega) = \dfrac{1}{j\omega + b} \Rightarrow X(f) = \dfrac{1}{j2\pi f + b}}$

CTFT Result … (Theory)

If we sample $x(t)$ at the rate of F_s samples/second – That is, sample every $T = 1/F_s$ sec – we get the <u>DT Signal</u> coming out of the ADC is:

$$x[n] = x(t)\,|_{t=nT} = x(nT)$$

For this example we get:

$$x[n] = \left[e^{-bt}u(t)\right]_{t=nT} = e^{-bTn}u[n]$$

$$= \left(e^{-bT}\right)^n u[n] \triangleq a^n u[n]$$

Note: $|a| < 1$

Now imagine that in theory we have all of the samples $x[n]$ $-\infty < n < \infty$ at the ADC output.

Then, in theory the DTFT$_\infty$ of this signal is found using the DTFT table to be:

B $$\boxed{X_\infty(\Omega) = \frac{1}{1 - ae^{-j\Omega}}}$$

DTFT$_\infty$ Result … (Theory)

For $|a| < 1$ which we have because:

$$a = e^{-bT} \ \ \& \ b > 0, \ T > 0$$

Now, in reality we can "collect" only $N < \infty$ samples in our computer:

$$x_N[n] = a^n, \ 0 \le n \le (N-1)$$

$$(\text{"Assume"} \ \ x_N[n] = 0 \ \ \text{elsewhere})$$

Necessary to connect the DFT result to the theoretical results we'd like to see.

The DTFT of this collected finite-duration is easily found "by hand":

C $$\boxed{X_N(\Omega) = \frac{1 - \left(ae^{-j\Omega}\right)^N}{1 - ae^{-j\Omega}}}$$

Note that we think of this as follows ("rectangular windowing"):

$$x_N[n] = x[n]w_N[n] \qquad w_N[n] = \begin{cases} 1, & 0, 1, 2, \ldots, N-1 \\ 0, & otherwise \end{cases}$$

… and DTFT theory tells us that

$$\underbrace{X_N(\Omega) = X_\infty(\Omega) * W_N(\Omega)}$$

A form of convolution (DT Freq. Domain Convolution)

… and this convolution has a "smearing" effect.

Finally, the DFT of the zero-padded collected samples is …

$$x_{zp}[n] = \left.\boxed{\begin{array}{l} x[0] \\ x[1] \\ \ldots \\ x[N-1] \\ 0 \\ \ldots \\ 0 \end{array}}\right\} \text{Total of } N_{zp} \text{ "points"}$$

$$\boxed{X_{zp}[k] = \sum_{n=0}^{N_{zp}-1} x_{zp}[n]e^{-j\frac{2\pi kn}{N_{zp}}}} \quad \boxed{\text{D}}$$

(The only part of this example we'd <u>really</u> "do")

Our theory tells us that the zero-padded DFT is nothing more than "points" on DTFT$_N$:

$$\boxed{X_{zp}[k] = X_N(\Omega_k)}$$

where $\Omega_k = \underbrace{\dfrac{2\pi k}{N_{zp}}}$ $\quad k = 0, 1, 2, \ldots, N_{zp} - 1$

Spacing between DFT "points" is $2\pi/N_{zp}$

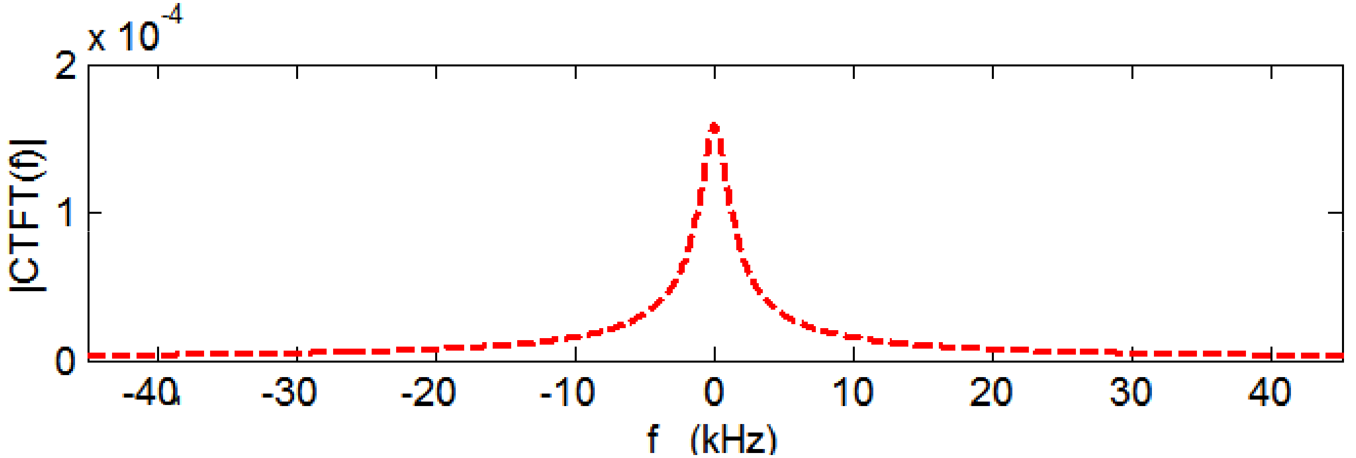
$\Rightarrow$ Increasing the amount of zero-padding gives closer spacing

Now run the m-file called DFT_Relations.m for different F_s , N, & N_{zp} values

Results from DFT_Relations.m

Plot #1: shows CTFT computed using: $X(f) = \dfrac{1}{j2\pi f + b}$ $\boxed{\text{A}}$

Notice that this is not ideally bandlimited, but is essentially bandlimited.



```
function DFT_Relations(Fs,N,N_zp)
%%% A routine showing CTFT, DTFT_inf, DTFT_N, and DFT Relations
%
% Inputs:  Fs = sampling rate in Hz
%          N = number of collected samples
%          N_zp = DFT size after zero-padding
%  Output:  Plots of CTFT, DTFT_inf, DTFT_N, and DFT

%%% Set parameters if user doesn't specify them
if nargin==0
   Fs=30e3;
   N=8;
   N_zp=8;
end

%%%% Compute the Theoretical CTFT
b=2*pi*1000;
f=-200000:100:200000;
CTFT=1./(j*2*pi*f + b);   %%% from table for x(t) = exp(-bt)u(t)
subplot(4,1,1)
h=plot(f/1e3,abs(CTFT),'r--');
```

Plot #2: shows DTFT$_\infty$ computed using:

(For plotting) $$X_\infty(\Omega) = \frac{1}{1-ae^{-j\Omega}} \qquad \boxed{\text{B}}$$

Our theory says that:

(For analysis) $$X_\infty(\Omega) = \frac{1}{T}\underbrace{\sum_{k=-\infty}^{\infty} X\left(\left(\frac{\Omega + k2\pi}{2\pi}\right)F_s\right)}$$

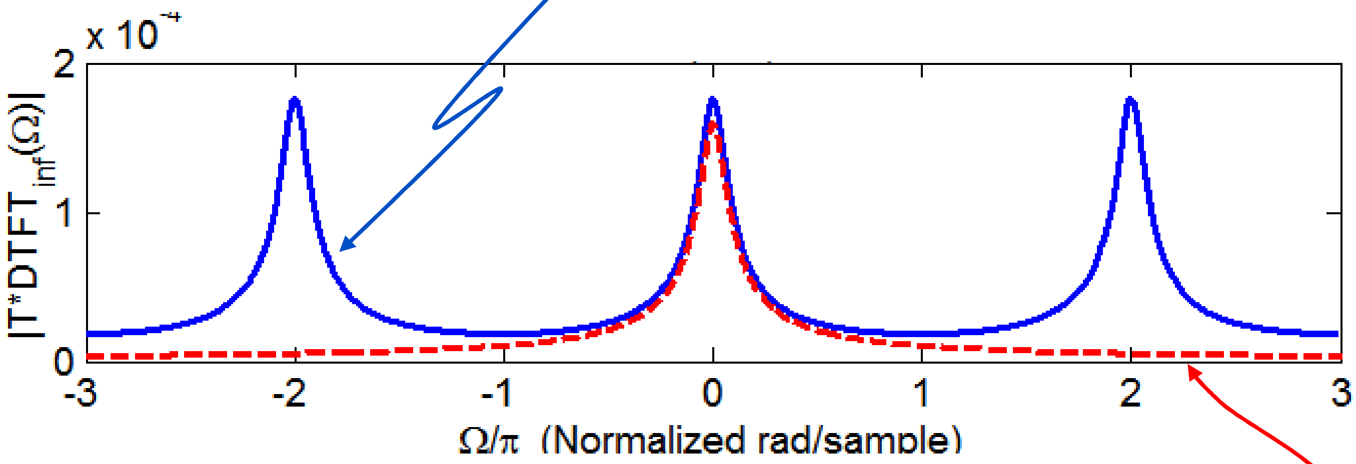
(CTFT rescaled to Ω and then shifted by multiples of 2π)

So we should see "replicas" in $X_\infty(\Omega)$ and we do!

We plot $TX_\infty(\Omega)$ to undo the 1/T here

Plot #2:

$$\text{DTFT}_\infty\text{:}\ \ X_\infty(\Omega) = \frac{1}{1-ae^{-j\Omega}}$$



We also plot the CTFT against $f \times \left(\dfrac{2\pi}{F_s}\right) = \Omega$

$$X(f) = \frac{1}{j2\pi f + b} \quad \Rightarrow \quad X(\Omega) = \frac{1}{j2\pi(\Omega \times F_s/2\pi) + b}$$

Recall the two equivalent axes:

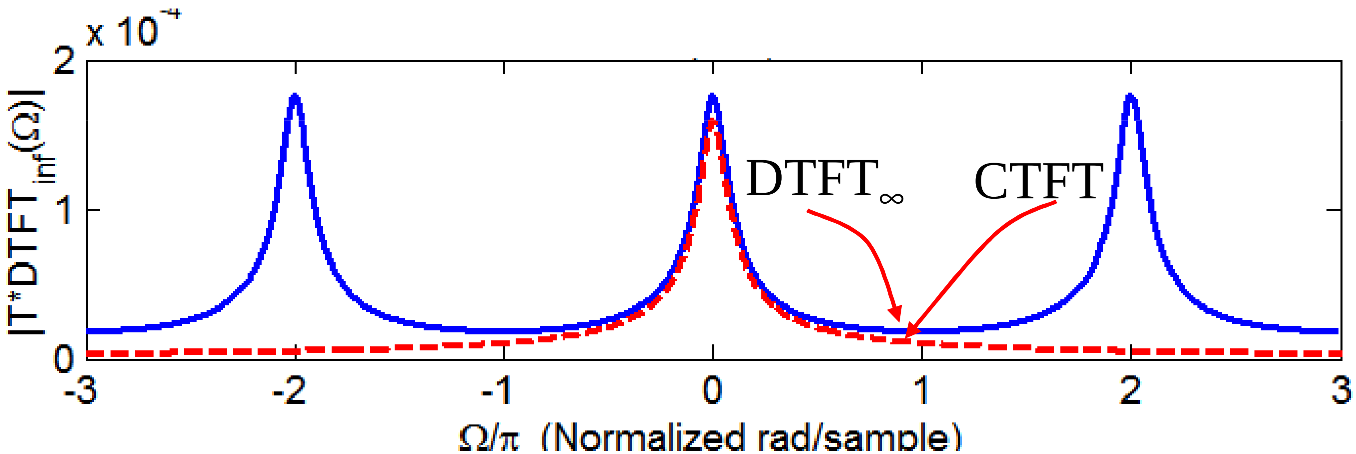
f	$-F_s$	$\dfrac{-F_s}{2}$	0	$\dfrac{F_s}{2}$	F_s
Ω	-2π	$-\pi$	0	π	2π

The theory in

$$X_{\infty}(\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X\left(\left(\frac{\Omega + k2\pi}{2\pi}\right)F_s\right)$$

says we'll see <u>significant</u> <u>aliasing</u> in $X_{\infty}(\Omega)$ unless F_s is high enough

The first error – visible in plot #2



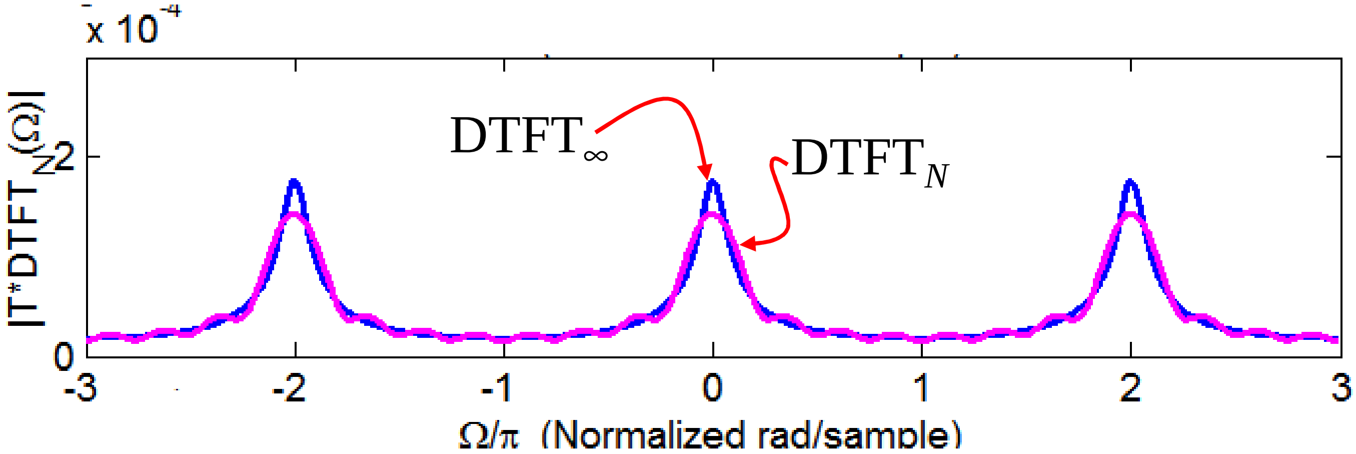
```
%%%%  Compute the Theoretical DTFT_inf
T=1/Fs;
a=exp(-b*T);   %%% computes exponential decay rate of sampled signal
omega=-3*pi:0.01:3*pi;
DTFT_inf=1./(1 - a*exp(-j*omega));   %%% from table for x[n] = a^n u[n]
subplot(4,1,2)
h=plot(omega/pi,abs(T*DTFT_inf));
set(h,'linewidth',2);
set(gca,'fontsize',12);
xlabel('\Omega/\pi  (Normalized rad/sample)')
ylabel('|T*DTFT_{inf}(\Omega)|')
hold on
h=plot(f/(Fs/2),abs(CTFT),'r--');
set(h,'linewidth',2);
axis_x([-3 3])
hold off
```

Plot #3 shows DTFT_N computed using $X_N(\Omega) = \dfrac{1-(ae^{-j\Omega})^N}{1-ae^{-j\Omega}}$ $\boxed{\text{C}}$

We see that $X_N(\Omega)$ shows signs of the "smearing" due to: $\boxed{X_N(\Omega) = X_\infty(\Omega) * W_N(\Omega)}$

Also called "leakage" error

The second error – visible in plot #3



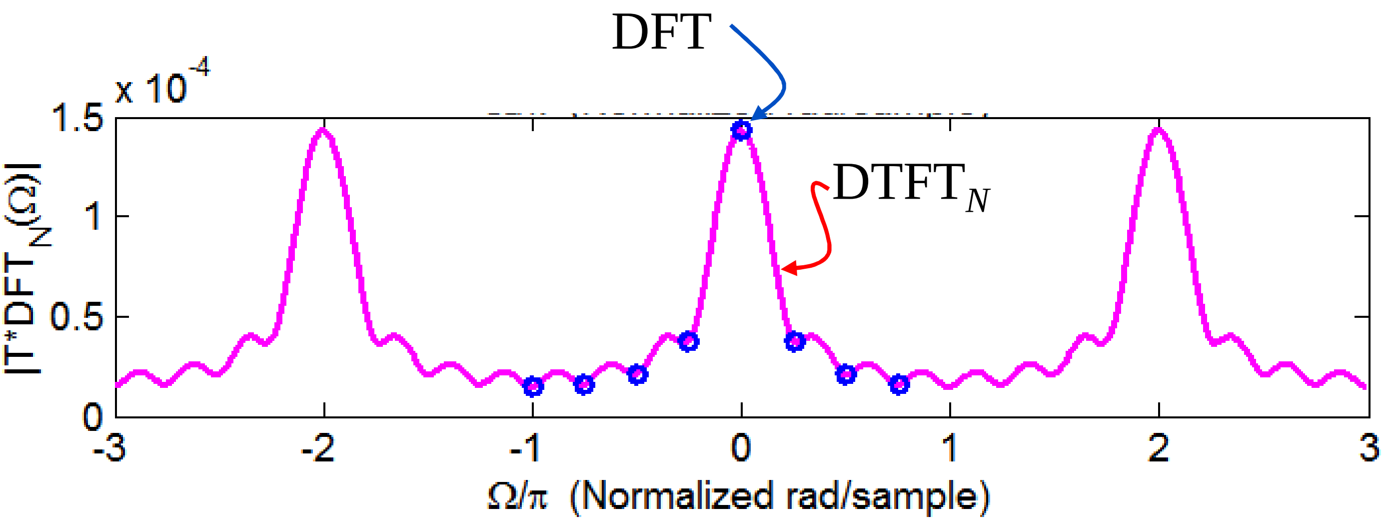
This "leakage" error is less significant as we increase *N*, the number of collected samples

```
%%%%  Compute the Theoretical DTFT_N
DTFT_N=(1-(a*exp(-j*omega)).^N)./(1 - a*exp(-j*omega));
            %%% from Eq. (7.6) in Kamen & Heck
subplot(4,1,3)
h=plot(omega/pi,abs(T*DTFT_inf));
set(h,'linewidth',2);
set(gca,'fontsize',12);
hold on
h=plot(omega/pi,abs(T*DTFT_N),'m');
set(h,'linewidth',2);
hold off
xlabel('\Omega/\pi  (Normalized rad/sample)')
ylabel('|T*DTFT_{N}(\Omega)|')
```

Plot #4 shows DFT computed using: $X_{zp}[k] = \sum_{n=0}^{N_{zp}-1} x_{zp}[n] e^{\frac{-j2\pi kn}{N_{zp}}}$ $\boxed{D}$

It is plotted vs. $\boxed{\Omega_k = \frac{2\pi k}{N_{zp}}}$ …but with the "right half" moved down to lie between $-\pi$ & 0 rad/sample

For comparison we also plot $X_N(\Omega)$



Note: We show an artificially small number of DFT points here

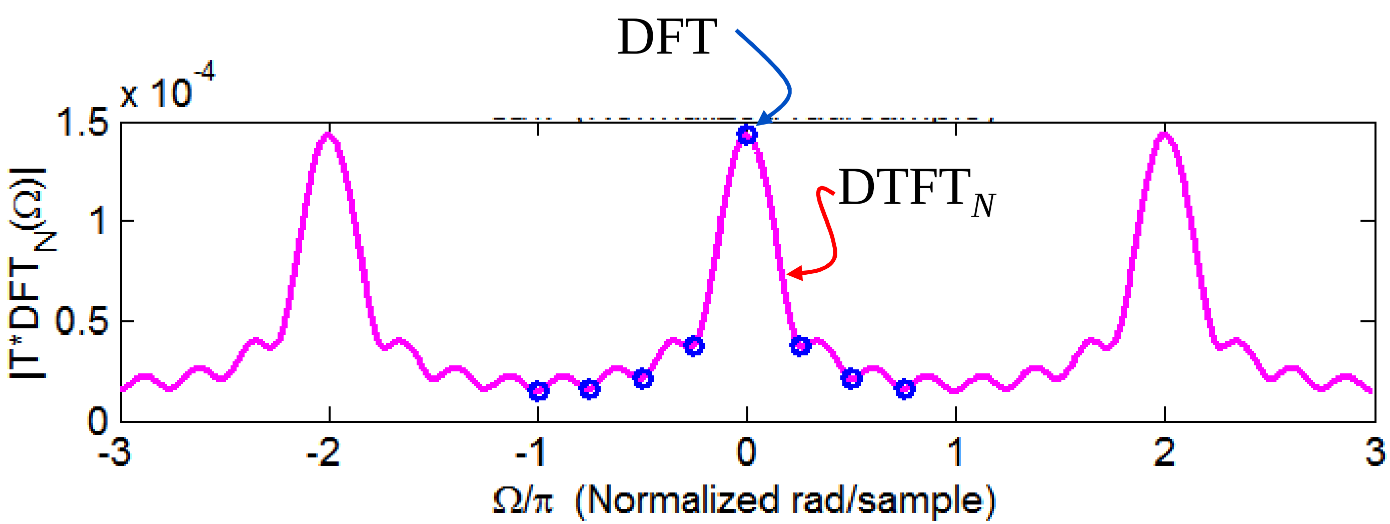
Theory says… $X_{zp}[k]$ points should lie on top of $X_N(\Omega)$… not $X_\infty(\Omega)$!!

We see that this is true

If N_{zp} is too small (i.e. $N_{zp} = N$) then there aren't enough "DFT points" on $X_N(\Omega)$ to allow us to see the real underlying shape of $X_N(\Omega)$

This is "Grid Error" and it is less significant when N_{zp} is large.

The third error – visible in plot #4



```
%%% Now do the DFT processing that could be done in practice on the N samples
n=0:(N-1);
nT=n*T;
x_N=exp(-b*nT);
%% Next command:
  %% fft computes fft with zero-padding to N_zp
  %% fftshift moves the points for [pi,2*pi) down to [-pi, 0) so we cover [-pi,pi)
DFT_N=fftshift(fft(x_N,N_zp));
%% now compute the frequencies where DFT points are
omega_k= ((-N_zp/2):((N_zp/2)-1))*pi/(N_zp/2);
subplot(4,1,4)
h=plot(omega/pi,abs(T*DTFT_N),'m');
set(h,'linewidth',2);
set(gca,'fontsize',12);
hold on
h=plot(omega_k/pi,abs(T*DFT_N),'bo');
set(h,'linewidth',2);
hold off
xlabel('\Omega/\pi  (Normalized rad/sample)')
ylabel('|T*DFT_{N}(\Omega)|')
```

Summary of Results: